

ON THE ERDŐS-FUCHS THEOREM

LI-XIA DAI AND HAO PAN

ABSTRACT. We prove several extensions of the Erdős-Fuchs theorem.

1. INTRODUCTION

The well-known Gauss circle conjecture says that

$$|\{(a, b) : a, b \in \mathbb{N}, a^2 + b^2 \leq n\}| = \frac{\pi}{4}n + O(n^{\frac{1}{4}+\epsilon}) \quad (1.1)$$

for any $\epsilon > 0$. The known best result due to Huxley is replacing $O(n^{\frac{1}{4}+\epsilon})$ by $O(n^{\frac{131}{416}})$. In general, for two non-empty subsets A, B of \mathbb{N} and $n \in \mathbb{N}$, define

$$r_{A,B}(n) := |\{(a, b) : a \in A, b \in B, a + b = n\}|.$$

Also, define

$$R_{A,B}(n) := \sum_{j \leq n} r_{A,B}(j),$$

i.e.,

$$R_{A,B}(n) = |\{(a, b) : a \in A, b \in B, a + b \leq n\}|.$$

Clearly (1.1) can be rewritten as

$$R_{\mathbb{N}^2, \mathbb{N}^2}(n) = \frac{\pi}{4}n + O(n^{\frac{1}{4}+\epsilon}),$$

where $\mathbb{N}^2 = \{a^2 : a \in \mathbb{N}\}$.

On the other hand, with help of the Fourier analysis, Hardy found that the remainder $O(n^{\frac{1}{4}+\epsilon})$ in (1.1) can't be replaced by $O(n^{\frac{1}{4}}(\log n)^{\frac{1}{4}})$. In 1956, for arbitrary non-empty infinite subset A of \mathbb{N} , Erdős and Fuchs [4] proved that as $n \rightarrow +\infty$,

$$R_{A,A}(n) = cn + o(n^{\frac{1}{4}}(\log n)^{-\frac{1}{2}}) \quad (1.2)$$

can't hold for any constant $c > 0$. This result is so-called the *Erdős-Fuchs theorem*. Subsequently, Jurkat (unpublished), and later Montgomery and Vaughan [9] showed that the $(\log n)^{-\frac{1}{2}}$ in the remainder term of (1.2) can be removed, i.e., it is impossible that

$$R_{A,A}(n) = cn + o(n^{\frac{1}{4}}), \quad n \rightarrow \infty \quad (1.3)$$

for some constant $c > 0$.

2010 *Mathematics Subject Classification*. Primary 11P70; Secondary 11B13, 11B34.

Key words and phrases. the Erdős-Fuchs theorem.

In [10], Sarközy considered the extension of the Erös-Fuchs theorem for the sum of are different subsets of \mathbb{N} . Let $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$ be two infinite subsets of \mathbb{N} . Suppose that for each $i \geq 1$, a_i is not very far from b_i , explicitly,

$$a_i - b_i = o(a_i^{\frac{1}{2}}(\log a_i)^{-1}). \quad (\text{S})$$

Then Sarközy proved that

$$R_{A,B}(n) = cn + o(n^{\frac{1}{4}}(\log n)^{-\frac{1}{2}}), \quad n \rightarrow +\infty \quad (1.4)$$

can not hold for any constant $c > 0$.

In [7], Horváth tried to remove the term $(\log n)^{-\frac{1}{2}}$ in the right side of (1.4). Define

$$A(n) := |\{a \in A : a \leq n\}|.$$

Under two assumptions

$$a_i - b_i = o(a_i^{\frac{1}{2}}), \quad i \rightarrow +\infty, \quad (\text{H1})$$

$$A(n) - B(n) = O(1), \quad n \geq 1, \quad (\text{H2})$$

Horváth proved that

$$R_{A,B}(n) = cn + o(n^{\frac{1}{4}}), \quad n \rightarrow +\infty \quad (1.5)$$

would not happen.

Notice that the assumption (H2), which says $A(n)$ and $B(n)$ are almost equal, seems a little too strong. So we wish to weaken the requirement for $A(n) - B(n)$, under the assumption that the difference $a_i - b_i$ is much smaller than $o(a_i^{\frac{1}{2}})$. In this paper, we shall give such a generalization of Horváth's result.

Theorem 1.1. *Suppose that $0 \leq \alpha \leq 1/4$. Let $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$ be two infinite subsets of \mathbb{N} satisfying that*

$$(1) \ a_i - b_i = o(a_i^{\frac{1}{2}-\alpha}) \text{ as } i \rightarrow +\infty;$$

$$(2) \ A(n) - B(n) = O(n^\alpha) \text{ for each } n \in \mathbb{N}.$$

Then

$$R_{A,B}(n) = cn + o(n^{\frac{1}{4}}), \quad n \rightarrow \infty \quad (1.6)$$

can not hold for any constant $c > 0$.

Note that $a_i - b_i = o(a_i^{\frac{1}{2}-\alpha})$ implies $A(n) - B(n) = o(n^{\frac{1}{2}-\alpha})$. Hence setting $\alpha = 1/4$, we get

Corollary 1.1.

$$R_{A,B}(n) = cn + o(n^{\frac{1}{4}}), \quad n \rightarrow \infty$$

can't hold for any constant $c > 0$, under the unique assumption

$$a_i - b_i = o(a_i^{\frac{1}{4}}).$$

We also can consider the generalizations of the Erdős-Fuchs theorem for the sums of more than two subsets of \mathbb{N} . Suppose that A_1, A_2, \dots, A_k are non-empty subsets of \mathbb{N} . Define

$$r_{A_1, \dots, A_k}(n) = |\{(a_1, a_2, \dots, a_k) : a_1 + \dots + a_k = n, a_1 \in A_1, \dots, a_k \in A_k\}|,$$

and define

$$R_{A_1, \dots, A_k}(n) = \sum_{j \leq n} r_{A_1, \dots, A_k}(j).$$

Horváth [5, 6] proved that for any $A \subseteq \mathbb{N}$,

$$\underbrace{R_{A, \dots, A}}_{k \text{ A's}}(n) = cn + o(n^{\frac{1}{4}}(\log n)^{-\frac{1}{2}}) \quad (1.7)$$

can't hold for any constant $c > 0$. Subsequently, Tang [11] obtained an extension of (1.3) for the sum of k A's, i.e., it is impossible that

$$\underbrace{R_{A, \dots, A}}_{k \text{ times}}(n) = cn + o(n^{\frac{1}{4}}). \quad (1.8)$$

Chen and Tang also proved a quantitative version of (1.8) in [2].

In [5, 6], Horváth factly considered $R_{A_1, \dots, A_k}(n)$. Assume that $A_1 = \{a_{1,1}, a_{1,2}, \dots\}$ and $A_2 = \{a_{2,1}, a_{2,2}, \dots\}$. Suppose that

$$a_{1,i} - a_{2,i} = o(a_{1,i}^{\frac{1}{2}}(\log a_{1,i})^{-\frac{k}{2}}), \quad i \rightarrow +\infty, \quad (\text{h1})$$

$$A_j(n) = \Theta(A_1(n)), \quad j = 3, \dots, k \quad (\text{h2})$$

for any sufficiently large n , where $f = \Theta(g)$ means $g \ll f \ll g$, i.e., there exist two constants $c_1, c_2 > 0$ such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for any sufficiently large n . Then Horváth [6] showed that for any constant $c > 0$,

$$R_{A_1, \dots, A_k}(n) = cn + o(n^{\frac{1}{4}}(\log n)^{1-\frac{3k}{4}}), \quad n \rightarrow +\infty, \quad (1.9)$$

is impossible. Under some additional assumptions, Tang [12] improved Horváth's result and showed that the remainder term can be reduced to $o(n^{\frac{1}{4}}(\log n)^{-\frac{1}{2}})$ or $o(n^{\frac{1}{4}}(\log n)^{-\frac{k+1}{2(k-1)}})$ according to whether k is even or odd.

Here we shall give an extension of (1.3) concerning $R_{A_1, \dots, A_k}(n)$.

Theorem 1.2. *Suppose that $0 < \beta \leq 1/2$ and $0 \leq \alpha \leq \beta/2$. Let A_1, A_2, \dots, A_k be some non-empty subsets of \mathbb{N} satisfying that*

- (1) $a_{1,i} - a_{2,i} = o(a_{1,i}^{\beta-\alpha})$ as $i \rightarrow +\infty$, where $a_{1,i}$ (resp. $a_{2,i}$) is the i -th elements of A_1 (resp. A_2);
- (2) $A_1(n) - A_2(n) = O(n^\alpha)$ for each $n \in \mathbb{N}$;
- (3) $A_1(n) = \Theta(n^\beta)$.

Then

$$R_{A_1, \dots, A_k}(n) = cn + o(n^{\frac{1}{4}}), \quad n \rightarrow \infty \quad (1.10)$$

can not hold for any constant $c > 0$.

Clearly the assumptions (2) and (3) of Theorem 1.2 also imply $A_2(n) = \Theta(n^\beta)$. Furthermore, if $A_j(n) = \Theta(A_1(n))$ for $j = 2, \dots, k$ and $R_{A_1, \dots, A_k}(n) = \Theta(n)$, then it is easy to verify that $A_1(n) = \Theta(n^{\frac{1}{k}})$. Hence (2) and (3) of Theorem 1.2 are valid under Horváth's assumption (h2).

In [1], Bateman showed that

$$\sum_{j \leq n} (R_{A,A}(j) - cn)^2 = o(n^{\frac{3}{2}}(\log n)^{-1}), \quad n \rightarrow +\infty \quad (1.11)$$

can't hold for any constant $c > 0$. Clearly the result of Bateman implies the Erdős-Fuchs theorem. In [3], Chen and Tang showed that for any constant $c > 0$, it is impossible that

$$\sum_{j \leq n} \left(\underbrace{R_{A, \dots, A}(j)}_{k \text{ times}} - cn \right)^2 = o(n^{\frac{3}{2}}), \quad n \rightarrow +\infty. \quad (1.12)$$

Now we can prove that

Theorem 1.3. (i) Under the assumptions of Theorem 1.1,

$$\sum_{j \leq n} (R_{A,B}(j) - cn)^2 = o(n^{\frac{3}{2}}), \quad n \rightarrow \infty \quad (1.13)$$

can not hold for any constant $c > 0$.

(ii) Under the assumptions of Theorem 1.2,

$$\sum_{j \leq n} (R_{A_1, \dots, A_k}(j) - cn)^2 = o(n^{\frac{3}{2}}), \quad n \rightarrow \infty \quad (1.14)$$

can not hold for any constant $c > 0$.

In Section 2, we shall establish an auxiliary lemma and use it to conclude the proof of Theorem 1.1. This lemma is also necessary to the proof of Theorem 1.2, which will be given in Section 3. Finally, we shall prove Theorem 1.3 in Section 4.

2. AN AUXILIARY LEMMA AND THE PROOF OF THEOREM 1.1

Lemma 2.1. Suppose that $0 < \beta \leq 1/2$ and $0 \leq \alpha \leq \beta/2$. Let $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$ be two infinite subsets of \mathbb{N} satisfying that

- (1) $a_i - b_i = o(a_i^{\beta-\alpha})$ for each $i \geq 1$;
- (2) $A(n) - B(n) = O(n^\alpha)$ for each $n \in \mathbb{N}$;
- (3) $A(n), B(n) \leq cn^\beta$ for each $n \in \mathbb{N}$, where $c > 0$ is a constant.

Then as $N \rightarrow +\infty$, we have

$$\sum_{n=0}^{\infty} \left(1 - \frac{1}{N}\right)^{2n} \cdot \left(\sum_{\substack{a \in A \\ a \leq n}} a - \sum_{\substack{b \in B \\ b \leq n}} b\right)^2 = o(N^{2+2\beta}), \quad (2.1)$$

and

$$\sum_{n=0}^{\infty} \left(1 - \frac{1}{N}\right)^{2n} \cdot (A(n) - B(n))^2 = o(N^{2\beta}). \quad (2.2)$$

Proof. For each $j \geq 1$, let the interval

$$\mathcal{I}_j = [\min\{a_j, b_j\}, \max\{a_j, b_j\} - 1].$$

Evidently $|\mathcal{I}_j| = o(a_j^{\beta-\alpha})$. Define

$$\lambda(n) = |\{j : n \in \mathcal{I}_j\}|.$$

Note that $n \in \mathcal{I}_j$ if and only if either $a_j \leq n < b_j$ or $b_j \leq n < a_j$. Hence

$$\lambda(n) = |A(n) - B(n)| = O(n^\alpha).$$

Thus

$$\begin{aligned} \left| \sum_{\substack{a \in A \\ a \leq n}} a - \sum_{\substack{b \in B \\ b \leq n}} b \right| &\leq \sum_{\substack{j \geq 1 \\ n \in \mathcal{I}_j}} \min\{a_j, b_j\} + \sum_{\substack{j \geq 1 \\ a_j, b_j \leq n}} |a_j - b_j| \\ &\leq \lambda(n) \cdot n + A(n) \cdot o(n^{\beta-\alpha}) = \lambda(n) \cdot n + o(n^{2\beta-\alpha}), \end{aligned}$$

where in the last step we used the assumption $A(n) \ll n^\beta$. It follows that

$$\sum_{n \leq x} \left(\sum_{\substack{a \in A \\ a \leq n}} a - \sum_{\substack{b \in B \\ b \leq n}} b \right)^2 \ll \sum_{n \leq x} (\lambda(n)^2 n^2 + o(n^{4\beta-2\alpha})) \ll x^2 \sum_{n \leq x} \lambda(n)^2 + o(x^{1+4\beta-2\alpha})$$

for any sufficiently large x . Define

$$\mathfrak{J}(x) = \max\{j \geq 1 : a_j \leq x \text{ or } b_j \leq x\}.$$

Clearly $\mathfrak{J}(x) \ll x^\beta$. We have

$$\sum_{n \leq x} \lambda(n)^2 \ll x^\alpha \sum_{n \leq x} \lambda(n) \leq x^\alpha \sum_{j \leq \mathfrak{J}(x)} |\mathcal{I}_j| = x^\alpha \cdot x^\beta \cdot o(x^{\beta-\alpha}) = o(x^{2\beta}),$$

as $x \rightarrow +\infty$, i.e.,

$$\sum_{n \leq x} \left(\sum_{\substack{a \in A \\ a \leq n}} a - \sum_{\substack{b \in B \\ b \leq n}} b \right)^2 = o(x^{2+2\beta}).$$

Now for any $\epsilon > 0$, there exists $x_0 = x_0(\epsilon) > 0$ such that if $x \geq x_0$, then

$$\sum_{n \leq x} \left(\sum_{\substack{a \in A \\ a \leq n}} a - \sum_{\substack{b \in B \\ b \leq n}} b \right)^2 \leq \epsilon x^{2+2\beta}.$$

Define

$$\psi(x) = \sum_{n \leq x} \left(\sum_{\substack{a \in A \\ a \leq n}} a - \sum_{\substack{b \in B \\ b \leq n}} b \right)^2.$$

Trivially, $\psi(x) = \sum_{n < x} (n \cdot n)^2 \leq x^5$ for any $x \geq 0$. Let $\rho = 1 - 1/N$. Applying the Stieltjes integral, we get

$$\begin{aligned} \sum_{n=0}^{\infty} \rho^{2n} \left(\sum_{\substack{a \in A \\ a \leq n}} a - \sum_{\substack{b \in B \\ b \leq n}} b \right)^2 &= \int_0^{\infty} \rho^{2x} d\psi(x) \\ &= \rho^{2x} \psi(x) \Big|_{x=0}^{+\infty} - 2 \int_0^{+\infty} \psi(x) \cdot \rho^{2x} \log \rho dx. \end{aligned}$$

Since $0 < \rho < 1$,

$$\lim_{x \rightarrow +\infty} \rho^{2x} \psi(x) \leq \lim_{x \rightarrow +\infty} \rho^{2x} x^{2+2\beta} = 0.$$

Let $\eta = -\log \rho$. Clearly $1/N \leq \eta \leq 2/N$. Then

$$\begin{aligned} & - \int_0^{+\infty} \psi(x) \cdot \rho^{2x} \log \rho dx = \eta \int_0^{+\infty} \psi(x) \cdot e^{-2\eta x} dx \\ &= \eta \int_{x_0}^{+\infty} \psi(x) \cdot e^{-2\eta x} dx + \eta \int_0^{x_0} \psi(x) \cdot e^{-2\eta x} dx \\ &\leq \eta \int_0^{+\infty} \epsilon x^{2+2\beta} \cdot e^{-2\eta x} dx + \eta \int_0^{x_0} x^5 dx \\ &= \frac{\Gamma(3+2\beta) \cdot \epsilon}{2^{3+2\beta} \eta^{2+2\beta}} + \frac{\eta x_0^6}{6} \leq \frac{\Gamma(3+2\beta) \cdot \epsilon N^{2+2\beta}}{2^{3+2\beta}} + \frac{x_0^6}{3N}, \end{aligned}$$

where Γ is the Gamma function. If $N \geq \epsilon^{-1} x_0^6$, we can get

$$\sum_{n=0}^{\infty} \rho^{2n} \left(\sum_{\substack{a \in A \\ a \leq n}} a - \sum_{\substack{b \in B \\ b \leq n}} b \right)^2 \leq 4\epsilon N^{2+2\beta}.$$

Since $\epsilon > 0$ can be arbitrarily small, we get (2.1).

Similarly, we have

$$\sum_{n \leq x} (A(n) - B(n))^2 = \sum_{n \leq x} \lambda(n)^2 \ll x^\alpha \sum_{n \leq x} \lambda(n) \leq x^\alpha \sum_{j \leq \mathfrak{J}(x)} |\mathcal{I}_j| = o(x^{2\beta}),$$

as $x \rightarrow +\infty$. For any $\epsilon > 0$, there exists $x_0 = x_0(\epsilon) > 0$ such that for any $x > x_0$,

$$\sum_{n \leq x} (A(n) - B(n))^2 \leq \epsilon x^{2\beta}.$$

Define

$$\phi(x) = \sum_{n \leq x} (A(n) - B(n))^2.$$

We also have

$$\begin{aligned} \sum_{n=0}^{\infty} \rho^{2n} (A(n) - B(n))^2 &= \rho^{2x} \phi(x) \Big|_{x=0}^{+\infty} - 2 \int_0^{+\infty} \phi(x) \cdot \rho^{2x} \log \rho dx \\ &= 2\eta \int_{x_0}^{+\infty} \phi(x) \cdot e^{-2\eta x} dx + 2\eta \int_0^{x_0} \phi(x) \cdot e^{-2\eta x} dx \\ &\leq 2\eta \int_0^{+\infty} \epsilon x^{2\beta} \cdot e^{-2\eta x} dx + 2\eta \int_0^{x_0} x^3 dx \\ &\leq \frac{\Gamma(1+2\beta) \cdot \epsilon N^{2\beta}}{2^{2\beta}} + \frac{x_0^4}{N} \leq 4\epsilon N^\beta \end{aligned}$$

provided that $N \geq \epsilon^{-1} x_0^4$. Thus (2.2) is concluded, too. \square

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Assume on the contrary that (1.6) is true. Define $\vartheta(n) = R_{A,B}(n) - cn$. Then $\vartheta(n) = o(n^{\frac{1}{4}})$. Furthermore, since

$$A(n)B(n) \leq R_{A,B}(2n) = 2cn + o(n^{\frac{1}{4}}),$$

we also have $A(n), B(n) \leq 2c^{\frac{1}{2}} n^{\frac{1}{2}}$ for the sufficiently large n .

For $|z| < 1$, let

$$F(z) = \sum_{a \in A} z^a, \quad G(z) = \sum_{b \in B} z^b.$$

Clearly

$$F(z)G(z) = \left(\sum_{a \in A} z^a \right) \cdot \left(\sum_{b \in B} z^b \right) = \sum_{n=0}^{\infty} z^n \sum_{\substack{a \in A, b \in B \\ a+b=n}} 1 = \sum_{n=0}^{\infty} r_{A,B}(n) z^n.$$

It follows that

$$\frac{F(z)G(z)}{1-z} = \left(\sum_{n=0}^{\infty} r_{A,B}(n) z^n \right) \cdot \left(\sum_{n=0}^{\infty} z^n \right) \sum_{n=0}^{\infty} z^n \sum_{j=0}^n r_{A,B}(j) = \sum_{n=0}^{\infty} R_{A,B}(n) z^n.$$

Thus

$$\begin{aligned} \frac{(F(z) + G(z))^2}{4(1-z)} &= \frac{F(z)G(z)}{1-z} + \frac{(F(z) - G(z))^2}{4(1-z)} \\ &= c \sum_{n=0}^{\infty} nz^n + \sum_{n=0}^n \vartheta(n)z^n + \frac{(F(z) - G(z))^2}{4(1-z)}, \end{aligned}$$

i.e.,

$$\frac{(F(z) + G(z))^2}{2} = \frac{2cz}{1-z} + 2(1-z) \sum_{n=0}^n \vartheta(n)z^n + \frac{(F(z) - G(z))^2}{2}.$$

Taking the derivative in z of both sides of the above equation, we get

$$\begin{aligned} (F'(z) + G'(z))(F(z) + G(z)) &= \frac{2c}{(1-z)^2} + (F'(z) - G'(z))(F(z) - G(z)) \\ &\quad + 2(1-z) \sum_{n=1}^n n\vartheta(n)z^{n-1} - 2 \sum_{n=0}^n \vartheta(n)z^n. \end{aligned} \quad (2.3)$$

Let m be a large integer to be chosen later. Let $\rho = 1 - 1/N$ and $z(\theta) = \rho e^{2\pi\sqrt{-1}\theta}$. For convenience, we abbreviate $z(\theta)$ as z . Clearly for any $n_1, n_2 \in \mathbb{N}$,

$$\int_0^1 z^{n_1} \cdot \bar{z}^{n_2} d\theta = \begin{cases} \rho^{2n_1}, & \text{if } n_1 = n_2, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$J = \int_0^1 |(F'(z) + G'(z))(F(z) + G(z))| \cdot \left| \frac{1 - z^m}{1 - z} \right|^2 d\theta, \quad (2.4)$$

$$J_1 = \int_0^1 \left| \frac{2c}{(1-z)^2} \right| \cdot \left| \frac{1 - z^m}{1 - z} \right|^2 d\theta, \quad (2.5)$$

$$J_2 = \int_0^1 \left| 2 \sum_{n=0}^{\infty} \vartheta(n)z^n \right| \cdot \left| \frac{1 - z^m}{1 - z} \right|^2 d\theta, \quad (2.6)$$

$$J_3 = \int_0^1 \left| 2(1-z) \sum_{n=0}^{\infty} (n+1)v(n+1)z^n \right| \cdot \left| \frac{1 - z^m}{1 - z} \right|^2 d\theta, \quad (2.7)$$

and

$$J_4 = \int_0^1 |(F'(z) - G'(z))(F(z) - G(z))| \cdot \left| \frac{1 - z^m}{1 - z} \right|^2 d\theta. \quad (2.8)$$

Evidently by (2.3), we have

$$J \leq J_1 + J_2 + J_3 + J_4.$$

In [7], Horváth showed that

$$J \gg mN^{\frac{3}{2}}, \quad J_1, J_2 \ll m^2N, \quad J_3 = o(m^{\frac{1}{2}}N^{\frac{7}{4}}).$$

We only need to give an upper bound for J_4 . By the Cauchy-Schwarz inequality,

$$\begin{aligned} J_4 &\leq 4 \int_0^1 \left| \frac{F'(z) - G'(z)}{1-z} \right| \cdot \left| \frac{F(z) - G(z)}{1-z} \right| d\theta \\ &\leq \frac{4}{\rho} \left(\int_0^1 \left| \frac{zF'(z) - zG'(z)}{1-z} \right|^2 d\theta \right)^{\frac{1}{2}} \left(\int_0^1 \left| \frac{F(z) - G(z)}{1-z} \right|^2 d\theta \right)^{\frac{1}{2}}. \end{aligned}$$

Note that

$$\frac{zF'(z) - zG'(z)}{1-z} = \frac{1}{1-z} \left(\sum_{a \in A} az^a - \sum_{b \in B} bz^b \right) = \sum_{n=0}^{\infty} z^n \left(\sum_{\substack{a \in A \\ a \leq n}} a - \sum_{\substack{b \in B \\ b \leq n}} b \right).$$

Applying (2.1) with $\beta = 1/2$, we have

$$\begin{aligned} &\int_0^1 \left| \frac{zF'(z) - zG'(z)}{1-z} \right|^2 d\theta \\ &= \int_0^1 \left(\sum_{n=0}^{\infty} z^n \left(\sum_{\substack{a \in A \\ a \leq n}} a - \sum_{\substack{b \in B \\ b \leq n}} b \right) \right) \left(\sum_{n=0}^{\infty} \bar{z}^n \left(\sum_{\substack{a \in A \\ a \leq n}} a - \sum_{\substack{b \in B \\ b \leq n}} b \right) \right) d\theta \\ &= \sum_{n=0}^{\infty} \rho^{2n} \left(\sum_{\substack{a \in A \\ a \leq n}} a - \sum_{\substack{b \in B \\ b \leq n}} b \right)^2 = o(N^3). \end{aligned}$$

Similarly, by (2.2),

$$\int_0^1 \left| \frac{F(z) - G(z)}{1-z} \right|^2 d\theta = \sum_{n=0}^{\infty} \rho^{2n} \left(\sum_{\substack{a \in A \\ a \leq n}} 1 - \sum_{\substack{b \in B \\ b \leq n}} 1 \right)^2 = o(N).$$

Thus $J_4 = o(N^2)$.

Since $J \leq J_1 + J_2 + J_3 + J_4$, there exists a constant $C > 1$ such that

$$mN^{\frac{3}{2}} \leq Cm^2N + o(m^{\frac{1}{2}}N^{\frac{7}{4}}) + o(N^2).$$

By letting $m = C^{-2}N^{\frac{1}{2}}$, we get an evident contradiction. \square

3. PROOF OF THEOREM 1.2

Let us turn to Theorem 1.2.

Lemma 3.1. *Suppose that $0 < \beta \leq 1/2$ and A_1, \dots, A_k are non-empty subsets of \mathbb{N} . Assume that $A_1(n), A_2(n) = \Theta(n^\beta)$ and $R_{A_1, \dots, A_k}(n) = \Theta(n)$. Then*

$$R_{A_3, \dots, A_k}(n) = \Theta(n^{1-2\beta}).$$

Proof. Evidently

$$\begin{aligned} R_{A_1, \dots, A_k}(n) &= \sum_{u=0}^n r_{A_1, \dots, A_k}(u) = \sum_{\substack{0 \leq v, w \leq n \\ v+w=n}} r_{A_1, A_2}(v) r_{A_3, \dots, A_k}(w) \\ &\leq \sum_{v=0}^n r_{A_1, A_2}(v) \sum_{w=0}^n r_{A_3, \dots, A_k}(w) \leq A_1(n) A_2(n) \sum_{w=0}^n r_{A_3, \dots, A_k}(w). \end{aligned}$$

Since $R_{A_1, \dots, A_k}(n) \gg n$ and $A_1(n), A_2(n) \ll n^\beta$, we get that

$$\sum_{w=0}^n r_{A_3, \dots, A_k}(w) \geq \frac{R_{A_1, \dots, A_k}(n)}{A_1(n) A_2(n)} \gg n^{1-2\beta}.$$

On the other hand, we also have

$$\begin{aligned} R_{A_1, \dots, A_k}(3n) &= \sum_{\substack{0 \leq v, w \leq 3n \\ v+w=3n}} r_{A_1, A_2}(v) r_{A_3, \dots, A_k}(w) \geq \sum_{v=0}^{2n} r_{A_1, A_2}(v) \sum_{w=0}^n r_{A_3, \dots, A_k}(w) \\ &\geq A_1(n) A_2(n) \sum_{w=0}^n r_{A_3, \dots, A_k}(w) \gg n^{2\beta} \sum_{w=0}^n r_{A_3, \dots, A_k}(w). \end{aligned}$$

It follows from $R_{A_1, \dots, A_k}(3n) \ll n$ that

$$\sum_{w=0}^n r_{A_3, \dots, A_k}(w) \ll n^{1-2\beta}.$$

□

Assume on the contrary that (1.10) holds. Let $\vartheta(n) = r_{A_1, \dots, A_k}(n) - cn$. Let

$$F_i(z) = \sum_{a \in A_i} z^a$$

for each $1 \leq i \leq k$. Then we have

$$\frac{F_1(z) F_2(z) \cdots F_k(z)}{1-z} = \sum_{n=0}^{\infty} r_{A_1, \dots, A_k}(n) z^n = \frac{cz}{(1-z)^2} + \sum_{n=0}^{\infty} \vartheta(n) z^n,$$

i.e.,

$$\begin{aligned} & (F_1(z) + F_2(z))^2 F_3(z) \cdots F_k(z) \\ &= \frac{4cz}{1-z} + 4(1-z) \sum_{n=0}^{\infty} \vartheta(n) z^n + (F_1(z) - F_2(z))^2 F_3(z) \cdots F_k(z). \end{aligned}$$

Taking the derivative in z , we obtain that

$$\begin{aligned}
& 2(F_1'(z) + F_2'(z))(F_1(z) + F_2(z)) \prod_{j=3}^k F_j(z) + (F_1(z) + F_2(z))^2 \sum_{j=3}^k F_j'(z) \prod_{\substack{3 \leq i \leq k \\ i \neq j}} F_i(z) \\
&= 2(F_1'(z) - F_2'(z))(F_1(z) - F_2(z)) \prod_{j=3}^k F_j(z) + (F_1(z) - F_2(z))^2 \sum_{j=3}^k F_j'(z) \prod_{\substack{3 \leq i \leq k \\ i \neq j}} F_i(z) \\
&+ \frac{4c}{(1-z)^2} + 4(1-z) \sum_{n=0}^{\infty} (n+1) \vartheta(n+1) z^n - 4 \sum_{n=0}^{\infty} \vartheta(n) z^n.
\end{aligned}$$

Let $\rho = 1 - 1/N$, $z = \rho e^{2\pi\sqrt{-1}\theta}$ and let m be a large integer to be chosen later. Let

$$\begin{aligned}
J &= \int_0^1 \left| 2(F_1'(z) + F_2'(z)) \prod_{j=3}^k F_j(z) + (F_1(z) + F_2(z)) \sum_{j=3}^k F_j'(z) \prod_{\substack{3 \leq i \leq k \\ i \neq j}} F_i(z) \right| \\
&\quad \cdot |F_1(z) + F_2(z)| \cdot \left| \frac{1 - z^m}{1 - z} \right|^2 d\theta,
\end{aligned}$$

and

$$\begin{aligned}
J_4 &= \int_0^1 \left| 2(F_1'(z) - F_2'(z)) \prod_{j=3}^k F_j(z) + (F_1(z) - F_2(z)) \sum_{j=3}^k F_j'(z) \prod_{\substack{3 \leq i \leq k \\ i \neq j}} F_i(z) \right| \\
&\quad \cdot |F_1(z) - F_2(z)| \cdot \left| \frac{1 - z^m}{1 - z} \right|^2 d\theta.
\end{aligned}$$

And let J_1, J_2, J_3 be the same ones in (2.5)-(2.7) respectively.

First, we shall give a lower bound of J . Let

$$G(z) := 2(F_1'(z) + F_2'(z)) \prod_{j=3}^k F_j(z) + (F_1(z) + F_2(z)) \sum_{j=3}^k F_j'(z) \prod_{\substack{3 \leq i \leq k \\ i \neq j}} F_i(z).$$

Write

$$G(z) = \sum_{n=0}^{\infty} g_n z^n, \quad (F_1'(z) + F_2'(z)) \prod_{j=3}^k F_j(z) = \sum_{n=0}^{\infty} h_n z^n.$$

Clearly $g_n \geq 2h_n \geq 0$ for each $n \geq 0$. Let \mathcal{A} denote the multiset $A_1 \cup A_2$, i.e., the common elements of A_1 and A_2 have the multiplicity 2 in \mathcal{A} . Then

$$F_1(z) + F_2(z) = \sum_{a \in A_1} z^a + \sum_{a \in A_2} z^a = \sum_{a \in \mathcal{A}} z^a.$$

Thus

$$\begin{aligned}
J &\geq \left| \int_0^1 \overline{G(z)} \cdot (F_1(z) + F_2(z)) \cdot \frac{1-z^m}{1-z} \cdot \frac{1-\bar{z}^m}{1-\bar{z}} d\theta \right| \\
&= \left| \int_0^1 \left(\sum_{n=0}^{\infty} g_n \bar{z}^n \right) \cdot \left(\sum_{a \in \mathcal{A}} z^a \right) \cdot \left(\sum_{n=0}^{m-1} z^n \right) \cdot \left(\sum_{n=0}^{m-1} \bar{z}^n \right) d\theta \right| \\
&= \sum_{\substack{a \in \mathcal{A}, u \geq 0 \\ 0 \leq v, w \leq m-1 \\ a+v=u+w}} \rho^{a+u+v+w} g_u \geq 2 \sum_{\substack{a \in \mathcal{A}, u \geq 0 \\ 0 \leq v, w \leq m-1 \\ a+v=u+w}} \rho^{a+u+v+w} h_u.
\end{aligned}$$

Note that

$$(F_1'(z) + F_2'(z)) \prod_{j=3}^k F_j(z) = \sum_{a \in \mathcal{A}} a z^{a-1} \cdot \sum_{n=0}^{\infty} r_{A_3, \dots, A_k}(n) z^n = \sum_{n=0}^{\infty} z^n \sum_{\substack{a \in \mathcal{A} \\ a \leq n+1}} a r_{A_3, \dots, A_k}(n-a+1).$$

It follows that

$$\begin{aligned}
\sum_{\substack{a \in \mathcal{A}, u \geq 0 \\ 0 \leq v, w \leq m-1 \\ a+v=u+w}} \rho^{a+u+v+w} h_u &= \sum_{\substack{a \in \mathcal{A}, u \geq 0 \\ 0 \leq v, w \leq m-1 \\ a+v=u+w}} \rho^{a+u+v+w} \sum_{\substack{b \in \mathcal{A} \\ b \leq u+1}} b r_{A_3, \dots, A_k}(u+1-b) \\
&\geq \sum_{\substack{a, b \in \mathcal{A}, b \leq u \\ 0 \leq v, w \leq m-1 \\ a+v=u+w}} \rho^{a+u+v+w} \cdot b r_{A_3, \dots, A_k}(u+1-b).
\end{aligned}$$

We may restrict the above summation to those a, b, u, v, w satisfying the following conditions:

- (1) $c_3 N \leq a \leq N$, where $c_3 = (c_1/(4c_2))^{\frac{1}{\beta}}$;
- (2) $b = a$, $a \leq u \leq N$;
- (3) $1 \leq w < m/2$, $w \leq v \leq w + m/2$.

Thus

$$\begin{aligned}
J &\geq 2 \sum_{\substack{a \in \mathcal{A}, c_3 N \leq a \leq N \\ 0 \leq w < m/2, w \leq v \leq w+m/2 \\ a \leq u \leq N, u-a=v-w}} \rho^{a+u+v+w} \cdot a r_{A_3, \dots, A_k}(u+1-a) \\
&\geq 2 \sum_{\substack{a \in \mathcal{A} \\ c_3 N \leq a \leq N}} a \rho^{a+(w-v+a)+v+w} \sum_{\substack{0 \leq w < m/2 \\ w \leq v \leq w+m/2}} r_{A_3, \dots, A_k}(v-w+1) \\
&\geq 2c_3 N \cdot \rho^{2N+2m} (\mathcal{A}(N) - \mathcal{A}(c_3 N)) \cdot \frac{m}{2} \sum_{j=0}^{m/2} r_{A_3, \dots, A_k}(j+1).
\end{aligned}$$

We have $\mathcal{A}(N) \geq A_1(N) \geq c_1 N^\beta$ and

$$\mathcal{A}(c_3 N) \leq A_1(c_3 N) + A_2(c_3 N) \leq 2c_2 c_3^\beta N^\beta = \frac{c_1}{2} N^\beta.$$

Furthermore, since $m \leq N$,

$$\rho^{2N+2m} \geq \left(1 - \frac{1}{N}\right)^{4N} \geq \frac{1}{2e^4}.$$

Hence by Lemma 3.1,

$$J \gg N^{1+\beta} m \sum_{j=0}^{m/2} r_{A_3, \dots, A_k}(j+1) \gg m^{2-2\beta} N^{1+\beta}.$$

Let us consider the upper bound J_4 . Clearly

$$\begin{aligned} J_4 \leq & 8 \int_0^1 \left| \frac{F_1'(z) - F_2'(z)}{1-z} \right| \cdot \left| \frac{F_1(z) - F_2(z)}{1-z} \right| \cdot \prod_{j=3}^k |F_j(z)| d\theta \\ & + 4 \sum_{j=3}^k \int_0^1 \left| \frac{F_1(z) - F_2(z)}{1-z} \right|^2 \cdot |F_j'(z)| \prod_{\substack{3 \leq i \leq k \\ i \neq j}} |F_i(z)| d\theta. \end{aligned}$$

Note that

$$|F_3(z) \cdots F_k(z)| \leq |F_3(\rho) \cdots F_k(\rho)| = \sum_{n=0}^{\infty} \rho^n \cdot r_{A_3, \dots, A_k}(n). \quad (3.1)$$

Let

$$\omega(x) = \sum_{n \leq x} r_{A_3, \dots, A_k}(n).$$

By Lemma 3.1, $\omega(x) \ll x^{1-2\beta}$. So letting $\eta = -\log \rho$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \rho^n \cdot r_{A_3, \dots, A_k}(n) &= \int_0^{+\infty} \rho^x d\omega(x) = \rho^x \omega(x) \Big|_0^{+\infty} - \log \rho \int_0^{+\infty} \omega(x) \cdot \rho^x dx \\ &\ll \eta \int_0^{+\infty} x^{1-2\beta} e^{-\eta x} dx = \frac{\Gamma(2-2\beta)}{\eta^{1-2\beta}} \ll N^{1-2\beta}. \end{aligned}$$

Thus by the Cauchy-Schwarz inequality,

$$\begin{aligned} & \int_0^1 \left| \frac{F_1'(z) - F_2'(z)}{1-z} \right| \cdot \left| \frac{F_1(z) - F_2(z)}{1-z} \right| \cdot \prod_{j=3}^k |F_j(z)| d\theta \\ & \ll N^{1-2\beta} \left(\int_0^1 \left| \frac{F_1'(z) - F_2'(z)}{1-z} \right|^2 d\theta \right)^{\frac{1}{2}} \cdot \left(\int_0^1 \left| \frac{F_1(z) - F_2(z)}{1-z} \right|^2 d\theta \right)^{\frac{1}{2}}. \end{aligned}$$

Applying Lemma 2.1, we obtain that

$$\begin{aligned} \int_0^1 \left| \frac{F'(z) - G'(z)}{1-z} \right|^2 d\theta &\leq \frac{1}{\rho} \int_0^1 \left| \frac{zF'(z) - zG'(z)}{1-z} \right|^2 d\theta \\ &= \frac{1}{\rho} \sum_{n=0}^{\infty} \rho^{2n} \left(\sum_{\substack{a_1 \in A_1 \\ a_1 \leq n}} a_1 - \sum_{\substack{a_2 \in A_2 \\ a_2 \leq n}} a_2 \right)^2 = o(N^{2+2\beta}), \end{aligned}$$

and

$$\int_0^1 \left| \frac{F(z) - G(z)}{1-z} \right|^2 d\theta = \sum_{n=0}^{\infty} \rho^{2n} \left(\sum_{\substack{a_1 \in A_1 \\ a_1 \leq n}} 1 - \sum_{\substack{a_2 \in A_2 \\ a_2 \leq n}} 1 \right)^2 = o(N^{2\beta}).$$

So

$$\int_0^1 \left| \frac{F'_1(z) - F'_2(z)}{1-z} \right| \cdot \left| \frac{F_1(z) - F_2(z)}{1-z} \right| \cdot \prod_{j=3}^k |F_j(z)| d\theta \ll N^{1-2\beta} \cdot o(N^{1+2\beta}) = o(N^2).$$

Similarly, for each $3 \leq j \leq k$,

$$\begin{aligned} |F'_j(z)| \prod_{\substack{3 \leq i \leq k \\ i \neq j}} |F_i(z)| &\leq |F'_j(\rho)| \prod_{\substack{3 \leq i \leq k \\ i \neq j}} |F_i(\rho)| = \sum_{a_3 \in A_3, \dots, a_k \in A_k} a_j \rho^{a_3 + \dots + a_k} \\ &= \sum_{n=0}^{\infty} \rho^n \sum_{\substack{a_3 \in A_3, \dots, a_k \in A_k \\ a_3 + \dots + a_k = n}} a_j \leq \sum_{n=0}^{\infty} \rho^n \cdot r_{A_3, \dots, A_k}(n) n. \end{aligned}$$

Now

$$\begin{aligned} \sum_{n=0}^{\infty} \rho^n n \cdot r_{A_3, \dots, A_k}(n) &= \int_0^{+\infty} \rho^x x d\omega(x) \\ &= \rho^x x \cdot \omega(x) \Big|_0^{+\infty} - \int_0^{+\infty} \omega(x) \cdot \rho^x (1 + x \log \rho) dx \\ &\ll \eta \int_0^{+\infty} x^{1-2\beta} e^{-\eta x} dx + \int_0^{+\infty} x^{1-2\beta} e^{-\eta x} dx \\ &= \frac{\Gamma(3-2\beta) + \Gamma(2-2\beta)}{\eta^{2-2\beta}} \ll N^{2-2\beta}. \end{aligned}$$

It follows that

$$\begin{aligned} \int_0^1 \left| \frac{F(z) - G(z)}{1-z} \right|^2 \cdot |F'_j(z)| \prod_{\substack{3 \leq i \leq k \\ i \neq j}} |F_i(z)| d\theta &\ll N^{2-2\beta} \int_0^1 \left| \frac{F(z) - G(z)}{1-z} \right|^2 d\theta \\ &= N^{2-2\beta} \cdot o(N^{2\beta}) = o(N^2). \end{aligned}$$

Thus we get

$$J_4 = o(N^2).$$

Recall that $J \leq J_1 + J_2 + J_3 + J_4$ and

$$J_1, J_2 \ll m^2 N, \quad J_3 = o(m^{\frac{1}{2}} N^{\frac{7}{4}}).$$

We may choose a large constant $C > 1$ such that

$$mN^{\frac{3}{2}} \leq Cm^2 N + o(m^{\frac{1}{2}} N^{\frac{7}{4}}) + o(N^2).$$

It immediately leads to a contradiction by setting $m = C^{-2} N^{\frac{1}{2}}$.

4. PROOF OF THEOREM 1.3

Here we only give the proof of (i) of Theorem 1.3, since the proof of (ii) is completely same.

Suppose that N is sufficiently large and $\rho = 1 - 1/N$. Let $\vartheta(n) = R_{A,B}(n) - cn$ and let J, J_1, J_2, J_3, J_4 be given by (2.4)-(2.8). We shall give the upper bounds of J_2, J_3 under the assumption

$$\varpi(x) := \sum_{n \leq x} \vartheta(n)^2 = o(x^{\frac{3}{2}}), \quad x \rightarrow +\infty. \quad (4.1)$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} J_2 &= \int_0^1 \left| \sum_{n=0}^{\infty} \vartheta(n) z^n \right| \cdot \left| \frac{1 - z^m}{1 - z} \right|^2 d\theta \\ &\leq \left(\int_0^1 \left| \sum_{n=0}^{\infty} \vartheta(n) z^n \right|^2 d\theta \right)^{\frac{1}{2}} \cdot \left(\int_0^1 \left| \frac{1 - z^m}{1 - z} \right|^4 d\theta \right)^{\frac{1}{2}}, \end{aligned}$$

where $z = \rho e^{2\pi\sqrt{-1}\theta}$. Clearly

$$\int_0^1 \left| \frac{1 - z^m}{1 - z} \right|^4 d\theta \leq \sum_{\substack{0 \leq a, b, c, d \leq m-1 \\ a+b=c+d}} 1 \leq m^3.$$

And

$$\int_0^1 \left| \sum_{n=0}^{\infty} \vartheta(n) z^n \right|^2 d\theta = \sum_{n=0}^{\infty} \vartheta(n)^2 \rho^{2n}.$$

Since $\varpi(x) = o(x^{\frac{3}{2}})$, for any $\epsilon > 0$, there exists $x_0 = x_0(\epsilon) > 0$ such that for any $x \geq x_0$, $\varpi(x) \leq \epsilon x^{\frac{3}{2}}$. Note that trivially $\varpi(x) \leq x^5$ for any $x \geq 0$. Letting

$\eta = -\log \rho$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \vartheta(n)^2 \rho^{2n} &= \int_0^{+\infty} \rho^{2x} d\varpi(x) = \varpi(x) \rho^{2x} \Big|_0^{+\infty} - 2 \log \rho \int_0^{+\infty} \varpi(x) \rho^{2x} dx \\ &\leq 2\eta \int_0^{+\infty} \epsilon x^{\frac{3}{2}} \cdot e^{2\eta x} dx + 2\eta \int_0^{x_0} x^5 dx \\ &\leq \frac{\epsilon \cdot \Gamma(\frac{5}{2})}{(2\eta)^{\frac{3}{2}}} + \frac{\eta x_0^6}{3} \leq 2\epsilon N^{\frac{3}{2}}, \end{aligned}$$

provided that $N \geq \epsilon^{-1} x_0^6$. So

$$\sum_{n=0}^{\infty} \vartheta(n)^2 \rho^{2n} = o(N^{\frac{3}{2}})$$

as $N \rightarrow +\infty$, and

$$J_2 = o(m^{\frac{3}{2}} N^{\frac{3}{4}}).$$

On the other hand, since $|1 - z^m| \leq 2$, we have

$$\begin{aligned} J_3 &= \int_0^1 \left| (1-z) \sum_{n=0}^{\infty} (n+1) \vartheta(n+1) z^n \right| \cdot \left| \frac{1-z^m}{1-z} \right|^2 d\theta \\ &\leq 2 \int_0^1 \left| \sum_{n=0}^{\infty} (n+1) \vartheta(n+1) z^n \right| \cdot \left| \frac{1-z^m}{1-z} \right| d\theta \\ &\leq 2 \left(\int_0^1 \left| \sum_{n=0}^{\infty} (n+1) \vartheta(n+1) z^n \right|^2 d\theta \right)^{\frac{1}{2}} \cdot \left(\int_0^1 \left| \frac{1-z^m}{1-z} \right|^2 d\theta \right)^{\frac{1}{2}}. \end{aligned}$$

Clearly

$$\int_0^1 \left| \frac{1-z^m}{1-z} \right|^2 d\theta \leq m.$$

And

$$\int_0^1 \left| \sum_{n=0}^{\infty} (n+1) \vartheta(n+1) z^n \right|^2 d\theta = \sum_{n=0}^{\infty} (n+1)^2 \vartheta(n+1)^2 \rho^{2n} \leq 4 \sum_{n=0}^{\infty} n^2 \vartheta(n)^2 \rho^{2n}.$$

Note that $(x^2 \rho^{2x})' = 2x \rho^{2x} + 2 \log \rho \cdot x^2 \rho^{2x}$. For any $\epsilon > 0$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} n^2 \vartheta(n)^2 \rho^{2n} &= \int_0^{+\infty} x^2 \rho^{2x} d\varpi(x) = -2 \int_0^{\infty} (x \rho^{2x} + \log \rho \cdot x^2 \rho^{2x}) \cdot \varpi(x) dx \\ &\leq 2\eta \int_0^{+\infty} \epsilon x^{\frac{3}{2}} \cdot x^2 e^{2\eta x} dx + 2 \int_0^{+\infty} \epsilon x^{\frac{3}{2}} \cdot x e^{2\eta x} dx + 2 \int_0^{x_0} (x + \eta \cdot x^2) x^5 dx \\ &= \frac{\epsilon \cdot (\Gamma(\frac{9}{2}) + 2\Gamma(\frac{7}{2}))}{(2\eta)^{\frac{7}{2}}} + \frac{2x_0^7}{7} + \frac{\eta x_0^8}{4} \leq 4N^{\frac{7}{2}}, \end{aligned}$$

provided that $N \geq \epsilon^{-1}x_0^8$. Hence

$$J_3 = o(m^{\frac{1}{2}}N^{\frac{7}{4}}).$$

Finally, for any $\epsilon > 0$, clearly

$$\sum_{n \leq N} \vartheta(n)^2 \leq \epsilon N^{\frac{3}{2}}$$

implies $\vartheta(n) \leq \epsilon^{\frac{1}{2}}N^{\frac{3}{4}}$. Thus we have

$$R_{A,B}(n) = cn + o(n^{\frac{3}{4}})$$

as $n \rightarrow +\infty$. Since $R_{A,B}(n) = \Psi(n)$, under the assumptions of Theorem 1.1, we know that $J \gg mN^{\frac{3}{2}}$, $J_4 = o(N^2)$ and $J_1 = O(m^2N)$. It follows from $J \leq J_1 + J_2 + J_3 + J_4$ that

$$mN^{\frac{3}{2}} \leq Cm^2N + o(m^{\frac{3}{2}}N^{\frac{3}{4}}) + o(m^{\frac{1}{2}}N^{\frac{7}{4}}) + o(m^{\frac{1}{2}}N^{\frac{7}{4}})$$

for some constant $C > 1$. Setting $m = C^{-2}N^{\frac{1}{2}}$, we immediately get a contradiction whenever N is sufficiently large. \square

REFERENCES

- [1] P. T. Bateman, *The Erdős-Fuchs theorem on the square of a power series*, J. Number Theory, **9**(1977), 330-337.
- [2] Y.-G. Chen and M. Tang, *A quantitative Erdős-Fuchs theorem and its generalization*, Acta Arith., **149**(2011), 171-180.
- [3] Y.-G. Chen and M. Tang, *A generalization of the classical circle problem*, Acta Arith., **152**(2012), 279-290.
- [4] P. Erdős and W. J. Fuchs, *On a problem of additive number theory*, J. London Math. Soc., **31**(1956), 67-73.
- [5] G. Horváth, *On a generalization of a theorem of Erdős and Fuchs*, Acta Math. Hungar., **92**(2001), 83-110.
- [6] G. Horváth, *On a theorem of Erdős and Fuchs*, Acta Arith., **103**(2002), 321-328.
- [7] G. Horváth, *An improvement of an extension of a theorem of Erdős and Fuchs*, Acta Math. Hungar., **104**(2004), 27-37.
- [8] M. N. Huxley, *Integer points, exponential sums and the Riemann zeta function*, in: *Number Theory for the Millennium, II*, Urbana, IL, 2000, A K Peters, Natick, MA, 2002, pp. 275-90.
- [9] H. L. Montgomery and R. C. Vaughan, *On the Erdős-Fuchs theorems*, A tribute to Paul Erdős, 331-338, Cambridge Univ. Press, Cambridge, 1990.
- [10] A. Sarközy, *On a theorem of Erdős and Fuchs*, Acta Arith., **37**(1980), 333-338.
- [11] M. Tang, *On a generalization of a theorem of Erdős and Fuchs*, Discrete Math., **309**(2009), 6288-6293.
- [12] M. Tang, *On a generalization of the Erdős-Fuchs theorem*, Int. J. Number Theory, **10**(2014), 955-961.

E-mail address: lilidainjnu@163.com

SCHOOL OF MATHEMATICAL SCIENCES, NANJING NORMAL UNIVERSITY, NANJING 210046,
PEOPLE'S REPUBLIC OF CHINA

E-mail address: `haopan79@zoho.com`

DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY, NANJING 210093, PEOPLE'S RE-
PUBLIC OF CHINA